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Efficient numerical method for two-dimensional phase change problems

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Abstract—The fine balance of the horizontal and vertical sweeps of the standard alternating direction implicit method is efficiently incorporated in the numerical scheme for the two-dimensional phase change problems. The present method isolates the non-linearity associated with the moving interface and accurately tracks the interface movement along both the coordinate axes. This avoids the need for iterations at ordinary nodes away from the interface. Numerical results are obtained and plotted for square and rectangular prisms undergoing phase change.

INTRODUCTION AND LITERATURE SURVEY

THE PHENOMENON of melting and solidification processes are widely encountered in many applications such as metal processing, food processing, water freezing, chemical processes, solar thermal energy systems, etc. The speciality in these processes lies in the fact that the interface movement is a time-dependent phenomenon. Due to this complexity, the number of available exact solutions is highly restricted and limited to a one-dimensional (1-D) semi-infinite region [1]. But in practice the geometry involved will be usually finite in length and multidimensional. Due to these factors the problem becomes all the more complex.

Numerical methods using enthalpy formulation [2–4] make an approximation that the phase change takes place over a range of temperatures. Due to this, these methods yield satisfactory results only when this range is small and occupies two to three nodal locations of the spatial grid. This requires the use of a finer grid which, in turn increases computing time. Further a fine spatial grid restricts the size of the maximum time step so that the scheme does not miss the latent heat contribution of some elements. The implicit finite-difference method of Meyer [5] and Shamsundar and Sparrow [6] resorts to nodal iterations for solution requiring large computing time.

The boundary fixing techniques used for two-dimensional (2-D) problems [7–9] yield complex equations and require iterative methods for solution. Extra computations were also needed for obtaining the solution with respect to the original space variables. Sparrow *et al.* [10] proposed a variable grid spacing near the interface to avoid nodal iterations. The method was cumbersome and permits only restricted interface movement over grid points.

Methods that track the moving interface continuously yield generally accurate results. By and large the earlier methods [11–13] used only explicit numerical schemes because of their simplicity. The restriction on the space–time ratio and the need for an accurate starting solution restricts their usage. When effective implicit methods are used [14, 15] they remove the restriction on the space–time ratio used. These methods yield a set of nonlinear algebraic equations involving the unknown nodal temperatures and interface location, which were evaluated by iteration at each nodal location thus requiring large computing times. Again the success of these methods depends upon the iterative technique used and this sometimes becomes cumbersome. Methods that track the interface continuously and avoid iterations at ordinary nodes away from the interface are restricted to only 1-D geometry [16, 17].

A survey of these methods suggests the need for a simple and accurate method for 2-D problems that tracks the interface continuously and does not require nodal iterations for solution. With this in aim, a numerical implicit scheme is proposed for 2-D phase change problems. The non-linearity associated with the interface is isolated by making an energy balance at the interface and the surrounding nodal locations. The method accurately tracks the interface along both the coordinate axes thereby eliminating the need for interpolating its location between coordinate axes. The energy equation is used implicitly in alternating directions similar to the conventional ADI method. This avoids the need for iterations at ordinary nodes away from the interface.

PROBLEM DESCRIPTION

Let the phase change medium, which is initially at a uniform temperature and above its phase change temperature, occupy a prism of square or rectangular cross-section. For $t > 0$, the sides of the prism are subjected to a constant temperature lower than its

NOMENCLATURE

a	convenient reference length
C	specific heat per unit volume
E, E', S, S'	elimination coefficients
F, F', G, G'	substitution coefficients
L	latent heat per unit volume
M, N	length of sides of quarter prism
p	node number
$[Q_p]_q$	$\frac{1+\eta}{2+\eta}T_{p-2,q}-\frac{2+\eta}{1+\eta}T_{p-1,q}+\frac{1-\eta}{2-\eta}T_{p+2,q}-\frac{2-\eta}{1-\eta}T_{p+1,q}$
$[Q_q]_p$	$\frac{1+\eta}{2+\eta}T_{p,q-2}-\frac{2+\eta}{1+\eta}T_{p,q-1}+\frac{1-\eta}{2-\eta}T_{p,q+2}-\frac{2-\eta}{1-\eta}T_{p,q+1}$
Ste	Stefan number, $C(u_f-u_0)/L$
t	dimensional time
T_s	non-dimensional temperature in solid region, $(u_s-u_f)/(u_f-u_0)$
T_l	non-dimensional temperature in liquid region, $(\alpha_l/\alpha_s)(u_l-u_f)/(u_f-u_0)$
u	dimensional temperature
$[W_p]_q$	$\frac{1}{2+\eta}T_{p-2,q}-\frac{1}{1+\eta}T_{p-1,q}$
$[W_q]_p$	$\frac{1}{2+\eta}T_{p,q-2}-\frac{1}{1+\eta}T_{p,q-1}$
x, y	dimensional distance
X	non-dimensional distance, x/a
$\Delta X, \Delta Y$	spatial increments
Y	non-dimensional distance, y/a
$[Z_p]_q$	$\frac{1}{2-\eta}T_{p+2,q}-\frac{1}{1-\eta}T_{p+1,q}$
$[Z_q]_p$	$\frac{1}{2-\eta}T_{p,q+2}-\frac{1}{1-\eta}T_{p,q+1}$.

Greek symbols

α	thermal diffusivity
ε	interface location in two dimensions
τ	non-dimensional time, $\alpha_s t/a^2$
$\Delta\tau$	time increment.

Subscripts

e	initial condition
f	evaluated at the interface
i, j	node index
l	liquid
$M, N, 0$	node index for the fixed wall
p, q	node index undergoing phase change
s	solid.

Superscripts

$', ''$	time level index.

phase change temperature. Solidification thus starts from all sides of the prism and proceeds into the medium. In terms of the non-dimensional parameters, the descriptive equations for the problem are

$$\frac{\partial T_m}{\partial \tau} = P_m\left(\frac{\partial^2 T_m}{\partial X^2}+\frac{\partial^2 T_m}{\partial Y^2}\right); \qquad m = s,l \tag{1}$$

where $P_s = 1$, $P_l = \alpha_l/\alpha_s$

$$\begin{aligned} T_l(X,Y,0) &= T_e \\ T_s(0,Y,\tau) &= T_s(2N,Y,\tau) = T_s(X,0,\tau) \\ &= T_s(X,2M,\tau) = -1 \\ T_s(X,\varepsilon(X,\tau),\tau) &= T_l(X,\varepsilon(X,\tau),\tau) = 0. \end{aligned} \tag{2}$$

Following the usual practice, the moving boundary equation is written in the form that contains spatial derivatives of temperature along one coordinate axis

$$\frac{\partial \varepsilon}{\partial \tau} = Ste\left[1+\left(\frac{\partial \varepsilon}{\partial X}\right)^2\right]\left[\frac{\partial T_s}{\partial Y}-\frac{\partial T_l}{\partial Y}\right] \quad \text{for} \quad Y = \varepsilon(X,\tau) \tag{3a}$$

$$\frac{\partial \varepsilon}{\partial \tau} = Ste\left[1+\left(\frac{\partial \varepsilon}{\partial Y}\right)^2\right]\left[\frac{\partial T_s}{\partial X}-\frac{\partial T_l}{\partial X}\right] \quad \text{for} \quad X = \varepsilon(Y,\tau). \tag{3b}$$

NUMERICAL METHOD

The symmetry of the problem permits the solution to be obtained for one quarter of the prism. Hence the interface is tracked in one quarter of the prism with surfaces at $X = N$ and $Y = M$ being treated as insulated. For convenience the analysis can be divided into three regions of study.

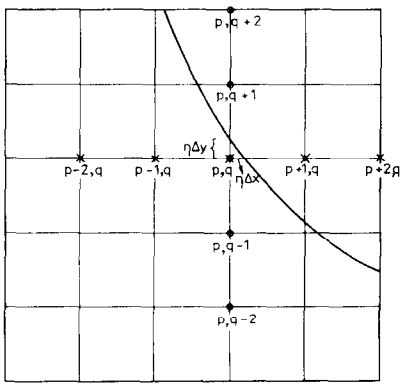

FIG. 1. Nodal points involved in alternating directions near interface.

Region near the interface

As shown in Fig. 1, let at any time τ the interface be near the nodal location (p, q) and let its distance measured along X and Y coordinates be $\eta_{p,q}\Delta X$ and $\bar{\eta}_{p,q}\Delta Y$ such that $-0.5 < \eta_{p,q} < 0.5$ and $-0.5 < \bar{\eta}_{p,q} < 0.5$. For simplicity the subscripts on η and $\bar{\eta}$ are dropped in the text that follows. The first- and second-order derivatives of the temperature are written using Taylor's series expansion

$$\left.\frac{\partial T}{\partial X}\right|_{\mathrm{f}} = \frac{1}{\Delta X}\left[\frac{1+\eta}{2+\eta}T_{p-2,q} - \frac{2+\eta}{1+\eta}T_{p-1,q}\right] + O(\Delta X)^2$$

$$\left.\frac{\partial T}{\partial X}\right|_{\mathrm{f}} = \frac{1}{\Delta X}\left[\frac{2-\eta}{1-\eta}T_{p+1,q} - \frac{1-\eta}{2-\eta}T_{p+2,q}\right] + O(\Delta X)^2$$

$$\left.\frac{\partial^2 T}{\partial X^2}\right|_{p-1,q} = \frac{2}{\Delta X^2}\left[\frac{1}{2+\eta}T_{p-2,q} - \frac{1}{1+\eta}T_{p-1,q}\right] + O(\Delta X)$$

$$\left.\frac{\partial^2 T}{\partial X^2}\right|_{p+1,q} = \frac{2}{\Delta X^2}\left[\frac{1}{2-\eta}T_{p+2,q} - \frac{1}{1-\eta}T_{p+1,q}\right] + O(\Delta X).$$

For tracking the interface movement along the two coordinate axes the moving boundary equation, equation (3), and the energy equation, equation (1), written in alternate directions, are expressed by using time-centered difference. In Fig. 1 the empty circles and crosses indicate the nodal points involved in writing the moving boundary equation at alternate time levels.

Interface along Y direction

$$\frac{\bar{\eta}' - \bar{\eta}}{\Delta\tau} = \frac{Ste}{2\Delta Y^2}\left[1+\left(\frac{4\varepsilon_{p+1,q} - 3\varepsilon_{p,q} - \varepsilon_{p+2,q}}{2\Delta X}\right)^2\right][Q_q]_p + \frac{Ste}{2\Delta Y^2}\left[1+\left(\frac{4\varepsilon'_{p+1,q} - 3\varepsilon'_{p,q} - \varepsilon'_{p+2,q}}{2\Delta X}\right)^2\right][Q'_q]_p. \quad (4)$$

Temperature implicit in Y direction.

$$\frac{T'_{p,q-1} - T_{p,q-1}}{\Delta\tau} = \frac{1}{\Delta Y^2}[W_q + W'_q]_p + \frac{1}{\Delta X^2}[T_{p-2,q-1} - 2T_{p-1,q-1} + T_{p,q-1}] \quad (5)$$

$$\frac{T'_{p,q+1} - T_{p,q+1}}{\Delta\tau} = \frac{\alpha_l/\alpha_s}{\Delta Y^2}[Z_q + Z'_q]_p + \frac{\alpha_l/\alpha_s}{\Delta X^2}[T_{p,q+1} - 2T_{p+1,q+1} + T_{p+2,q+1}]. \quad (6)$$

Interface along X direction

$$\frac{\eta'' - \eta'}{\Delta\tau} = \frac{Ste}{2\Delta X^2}\left[1+\left(\frac{4\varepsilon'_{p,q+1} - 3\varepsilon'_{p,q} - \varepsilon'_{p,q+2}}{2\Delta Y}\right)^2\right][Q'_p]_q + \frac{Ste}{2\Delta X^2}\left[1+\left(\frac{4\varepsilon''_{p,q+1} - 3\varepsilon''_{p,q} - \varepsilon''_{p,q+2}}{2\Delta Y}\right)^2\right][Q''_p]_q. \quad (7)$$

Temperature implicit in X direction.

$$\frac{T''_{p-1,q} - T'_{p-1,q}}{\Delta\tau} = \frac{2}{\Delta Y^2}[W'_q]_p + \frac{1}{2\Delta X^2}\times[(T''_{p-3,q} - 2T''_{p-2,q} + T''_{p-1,q}) + (T'_{p-3,q} - 2T'_{p-2,q} + T'_{p-1,q})] \quad (8)$$

$$\frac{T''_{p+1,q} - T'_{p+1,q}}{\Delta\tau} = \frac{2\alpha_l/\alpha_s}{\Delta Y^2}[Z'_q]_p + \frac{\alpha_l/\alpha_s}{2\Delta X^2}\times[(T''_{p+1,q} - 2T''_{p+2,q} + T''_{p+3,q}) + (T'_{p+1,q} - 2T'_{p+2,q} + T'_{p+3,q})]. \quad (9)$$

Equations (4)–(6) and equations (7)–(9) are alternately solved for the three unknowns η', $T'_{p,q-1}$, $T'_{p,q+1}$ and η'', $T''_{p-1,q}$, $T''_{p+1,q}$, respectively. The unknown nodal temperatures $T'_{p,q-2}$, $T'_{p,q+2}$, $T''_{p-3,q}$, $T''_{p-2,q}$, $T''_{p+2,q}$ and $T''_{p+3,q}$ are expressed in terms of the above three unknowns by the use of the Gaussian elimination technique used in alternate directions and described below.

Using centred time difference for the spatial derivatives of Y, energy equation (1) is written as follows.

At time step τ'

$$\frac{T'_{i,j} - T_{i,j}}{\Delta\tau} = \frac{P_m}{\Delta X^2}[T_{i-1,j} - 2T_{i,j} + T_{i+1,j}] + \frac{P_m}{2\Delta Y^2}[(T'_{i,j-1} - 2T'_{i,j} + T'_{i,j+1}) + (T_{i,j-1} - 2T_{i,j} + T_{i,j+1})]$$

which is rearranged as

$$-\beta T'_{i,j-1} + (1+2\beta)T'_{i,j} - \beta T'_{i,j+1} = \mathrm{RHSY}_{i,j} \quad (10)$$

where

$$\mathrm{RHSY}_{i,j} = T_{i,j} + \beta(T_{i,j-1} - 6T_{i,j} + T_{i,j+1} + 2T_{i-1,j} + 2T_{i+1,j})$$

and

$$\beta = \frac{P_m\Delta\tau}{2\Delta X^2} = \frac{P_m\Delta\tau}{2\Delta Y^2}.$$

At time step τ''

$$\frac{T''_{i,j}-T'_{i,j}}{\Delta\tau}=\frac{P_m}{2(\Delta X)^2}[(T''_{i-1,j}-2T''_{i,j}+T''_{i+1,j})+(T'_{i-1,j}-2T'_{i,j}+T'_{i+1,j})]+\frac{P_m}{(\Delta Y)^2}[T'_{i,j-1}-2T'_{i,j}+T'_{i,j+1}]$$

which is rearranged as

$$-\beta T''_{i-1,j}+(1+2\beta)T''_{i,j}-\beta T''_{i+1,j}=\mathrm{RHS}X_{i,j} \quad (11)$$

where

$$\mathrm{RHS}X_{i,j}=T'_{i,j}+\beta(T'_{i-1,j}-6T'_{i,j}+T'_{i+1,j}+2T'_{i,j-1}+2T'_{i,j+1}).$$

Solid region

At time step τ'. With boundary temperature at nodal point $(i,0)$ prescribed, the forward elimination yields

$$E_{i,0}=T'_{i,0}\ \text{(prescribed)}$$

$$S_{i,0}=0$$

$$E_{i,j}=(\mathrm{RHS}Y_{i,j}+\beta E_{i,j-1})/D \quad (12a)$$

$$S_{i,j}=\beta/D \quad (12b)$$

where $D=1+2\beta-\beta S_{i,j-1}$ and evaluated from $j=1$ to $q-2$. With nodal temperature $T'_{i,q-1}$ evaluated (explained later) the unknown nodal temperatures at nodal locations $i<j<q-2$ are found from the backward substitution equation

$$T'_{i,j}=S_{i,j}T'_{i,j+1}+E_{i,j} \quad (13)$$

and $j=q-2$ to 1.

At time step τ''. Forward elimination yields

$$E_{0,j}=T''_{0,j}\ \text{(prescribed)}$$

$$S_{0,j}=0$$

$$E_{i,j}=(\mathrm{RHS}X_{i,j}+\beta E_{i-1,j})/D \quad (14a)$$

$$S_{i,j}=\beta/D \quad (14b)$$

where $D=1+2\beta-\beta S_{i-1,j}$ and evaluated from $i=1$ to $p-2$. With nodal temperature $T''_{p-1,q}$ evaluated (explained later) the unknown nodal temperatures at nodal locations $1<i<p-2$ are evaluated from the backward substitution equation

$$T''_{i,j}=S_{i,j}T''_{i+1,j}+E_{i,j} \quad (15)$$

and $i=p-2$ to 1.

Liquid region

At time step τ'. With nodal temperature, $T_{i,M}$ evaluated (explained later), the backward elimination yields

$$E'_{i,M}=T'_{i,M}\ \text{(evaluated)}$$

$$S'_{i,M}=0$$

$$E'_{i,j}=(\mathrm{RHS}Y_{i,j}+\beta E'_{i,j+1})/D' \quad (16a)$$

$$S'_{i,j}=\beta/D' \quad (16b)$$

where $D'=1+2\beta-\beta S'_{i,j+1}$ and evaluated from $j=M-1$ to $q+2$. With nodal temperature $T'_{p,q+1}$ evaluated, the unknown nodal temperatures are evaluated from the forward substitution equation

$$T'_{i,j}=S'_{i,j}T'_{i,j-1}+E'_{i,j} \quad (17)$$

and $j=q+2$ to $M-1$.

At time step τ''.

$$E'_{N,j}=T''_{N,j}\ \text{(evaluated)}$$

$$S'_{N,j}=0$$

$$E'_{i,j}=(\mathrm{RHS}X_{i,j}+\beta E'_{i+1,j})/D' \quad (18a)$$

$$S'_{i,j}=\beta/D' \quad (18b)$$

where $D'=1+2\beta-\beta S'_{i+1,j}$ and evaluated from $i=N-1$ to $p+2$ and the nodal temperatures obtained from the forward substitution equation

$$T''_{i,j}=S'_{i,j}T''_{i-1,j}+E'_{i,j} \quad (19)$$

and $i=p+2$ to $N-1$.

Since no nodal temperatures are prescribed in this liquid region, a combined use of the forward and backward elimination technique is used to reduce the number of unknowns in equations (4)–(9).

At time step τ'. The backward elimination yields

$$S'_{i,M}=0$$

$$S'_{i,j}=\beta/D' \quad (20)$$

where $D'=1+2\beta-\beta S'_{i,j+1}$ evaluated from $j=M-1$ to $q+2$.

The unknown nodal temperatures near the interface are given by equation

$$T'_{i,q+2}=S'_{i,q+2}T'_{i,q+1}+F'T'_{i,M}+G' \quad (21)$$

where the coefficients F', G' are evaluated from the forward substitution equation

$$G'=0$$

$$F'=1$$

$$G'=G'+F'\mathrm{RHS}Y_{i,j}/D' \quad (22a)$$

$$F'=F'S'_{i,j} \quad (22b)$$

where $D'=1+2\beta-\beta S'_{i,j+1}$ and evaluated from $j=q+2$ to $M-1$.

Similarly the forward elimination yields

$$S_{i,q+1}=0$$

$$S_{i,j}=\beta/D \quad (23)$$

where $D=1+2\beta-\beta S_{i,j-1}$ and evaluated from $j=q+2$ to $M-1$.

The unknown nodal temperatures near the insulated surface are given by equation

$$T'_{i,M-1}=S_{i,M-1}T'_{i,M}+FT'_{i,q+1}+G \quad (24)$$

where F and G are evaluated from the backward

substitution equation

$$G = 0$$

$$F = 1$$

$$G = G + \mathrm{RHS}Y_{i,j}/D \tag{25a}$$

$$F = FS_{i,j} \tag{25b}$$

where $D = 1+2\beta-\beta S_{i,j-1}$ and evaluated from $j = M-1$ to $q+2$.

The number of unknowns in equation (24) can be reduced by making use of the adiabatic condition and the resulting equation is written as

$$(1+2\beta-2\beta S_{i,M-1})T'_{i,M} - 2\beta F T'_{i,q+1} = 2\beta G + \mathrm{RHS}Y_{i,M}. \tag{26}$$

For the planes $1 < i < N-1$ and $1 < j < M-1$ that do not intersect the interface and thereby lie totally in the solid region, the nodal temperatures are obtained by using equation (13) evaluated from $M-1$ to 1 and equation (15) for $i = N-1$ to 1. Use of these equations requires the nodal temperatures on the insulated boundaries, i.e. $T'_{i,M}$ and $T''_{N,j}$. These are obtained from the insulated boundary condition as

$$T'_{i,M} = \frac{\mathrm{RHS}Y_{i,M}+2\beta E_{i,M-1}}{1+2\beta-2\beta S_{i,M-1}} \tag{27}$$

and

$$T''_{N,j} = \frac{\mathrm{RHS}X_{N,j}+2\beta E_{N-1,j}}{1+2\beta-2\beta S_{N-1,j}}. \tag{28}$$

So far a set of equations are developed that are needed in evaluating the unknown interface location and the nodal temperatures. In using these equations the following procedure is followed.

Computational procedure

The order in which the foregoing equations are used in computations is explained here.

At time level τ'. Computations are started from the adiabatic surface of nodal plane $i = N$ and terminated with the nodal plane $i = 1$. The interface location along the nodal planes that intersect the interface is obtained by solving equations (4)–(6) simultaneously with equation (26). The temperature profile will be symmetric about the nodal plane $i = N$ as well as the shape of the interface. Hence the spatial derivatives of these profiles about the plane $i = N$ becomes zero. Use of equations (14) and (21) in equations (4)–(6) reduces the number of unknowns to four which are η', $T'_{p,q-1}$, $T'_{p,q+1}$ and $T'_{p,M}$. Equations (4)–(6) can be solved simultaneously with equation (26) using any convenient method. In this work the method of steepest descent is used and the convergence was obtained within four or five iterations.

After the successful evaluation of the interface location and the adjacent nodal temperatures on the nodal plane $i = N$, the unknown nodal temperatures in the solid and liquid regions away from the interface along the plane $i = N$ are obtained without nodal iterations from equations (13) and (17), respectively.

At nodal planes $i = N-1$ to 1, the terms representing the slope of the interface contains known values, because of the backward representation used for the slope. For those nodal planes that do not intersect the interface, the nodal temperatures are obtained directly from equation (18). At this juncture all the nodal temperatures in the region are known. The location of the part of the interface that lies parallel to the Y axis can be easily extracted from equation (7) written for this time step.

At time level τ''. Now the computations are started from the other adiabatic plane $j = M$ and terminated at $j = 1$. On the plane $j = M$ again the symmetry condition is used. The nodal temperatures $T''_{p-3,q}$, $T''_{p-2,q}$ that appear as unknowns in equation (8) can be expressed in terms of $T''_{p-1,q}$ by the use of equation (15). It is also to be noted that equation (8) yields explicitly the nodal temperature $T''_{p-1,M}$. Similarly equation (9) also yields explicitly the nodal temperature $T''_{p+1,M}$ on the liquid side if $T''_{p+1,M-1}$ is assumed to be equal to $T''_{p+1,M}$. The error due to this approximation will be negligible as it is confined to one nodal point near the interface in the liquid region. But this gives considerable advantage in saving computing time.

Since the two nodal temperatures $T''_{p-1,q}$ and $T''_{p+1,q}$ are explicitly obtained from equations (8) and (9), respectively, the interface location can be determined by solving only equation (7). The unknown nodal temperatures in the solid and liquid regions are obtained from equations (15), (19) and (28) thereby avoiding costly nodal iterations.

After evaluating all the nodal temperatures, the interface location that lies parallel to the X side is obtained from equation (4) written for this time step. The computations for the next time step are started from the plane $j = M$ thereby maintaining the balance between the horizontal and vertical sweeps of the method.

During the initial periods of solidification when the interface is within $1\frac{1}{2}$ times the spatial distance from the wall, the temperature profile in the solid region is approximated to be linear. Thus

$$[Q_q]_p \equiv \frac{1}{p+\eta} + \frac{1-\eta}{2-\eta} T_{p+2,q} - \frac{2-\eta}{1-\eta} T_{p+1,q}.$$

At time $\tau = 0$, the singularity is eliminated by taking $Q \equiv 0$ in equations (4) and (7).

Finally the temperature $T'_{p,q}$ is evaluated using a three point Lagrangian interpolation equation given as

$$T'_{p,q} = -\frac{2\eta'}{1-\eta'} T'_{p+1,q} + \frac{\eta'}{2-\eta'} T''_{p+2,q} \quad \text{for} \quad -0.5 < \eta' < 0$$

$$T'_{p,q} = \frac{2\eta'}{1+\eta'} T'_{p-1,q} - \frac{\eta'}{2+\eta'} T''_{p-2,q} \quad \text{for} \quad 0 < \eta' < +0.5.$$

NUMERICAL RESULTS

Numerical results are obtained for phase change in square and rectangular prisms. The quarter is divided by a uniform 11×11 grid in the former case and 11×8 in the later. Figures 2–6 indicate the results obtained for a square prism subjected to equal temperature on all sides. Figure 2 shows the interface location plotted for various times for $Ste = 1/1.5613$ and $T_e = 0$. It can be seen that the location of the interface along the adiabatic surfaces are quite symmetrical for all times

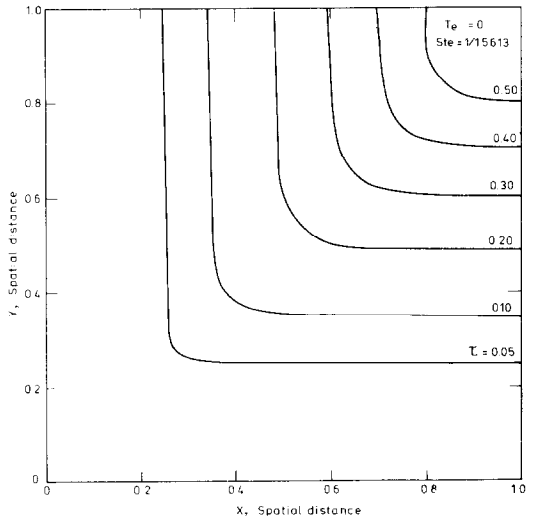

FIG. 2. Interface location at various times—square prism.

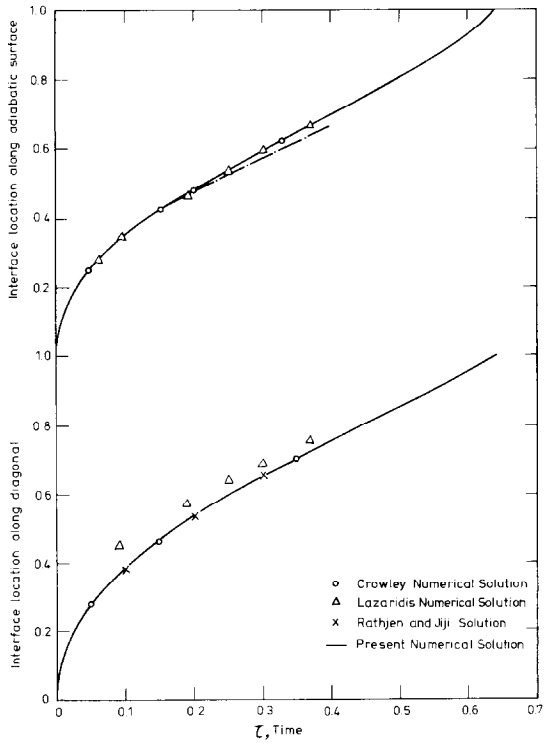

FIG. 3. Interface location along the diagonal and adiabatic surfaces.

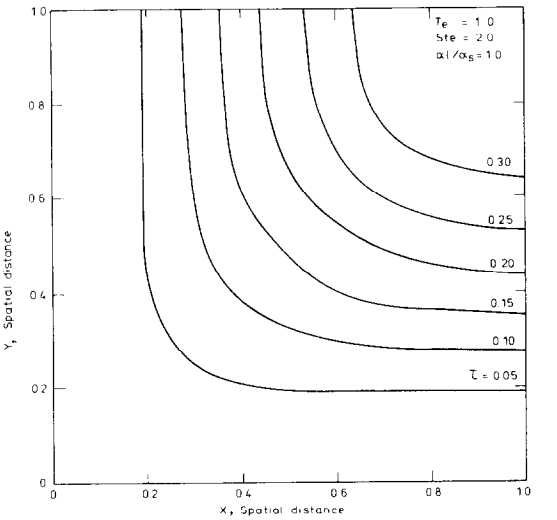

FIG. 4. Interface location at various times—square prism.

indicating a perfect balance of the horizontal and vertical sweeps of the method. During the initial periods of solidification the interface remains squarer in shape indicating principally 1-D heat flow. As time progresses, the curve on the diagonal gradually flattens indicating the 2-D effect. The prism totally solidifies at $\tau = 0.63$.

The interface locations along the adiabatic surface and along the diagonal of prism are plotted in Fig. 3. The dotted line represents the Stefan solution for the corresponding 1-D problem. The present numerical values agree well with refs. [3, 13]. When a comparison is drawn for the interface location along the diagonal of the prism, Lazaridis values fall above the present values

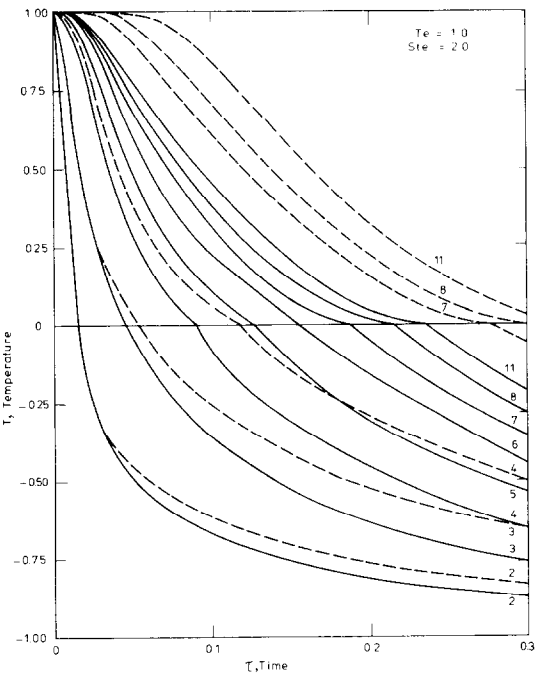

FIG. 5. Temperature histories along the mid and adiabatic planes of the quarter prism.

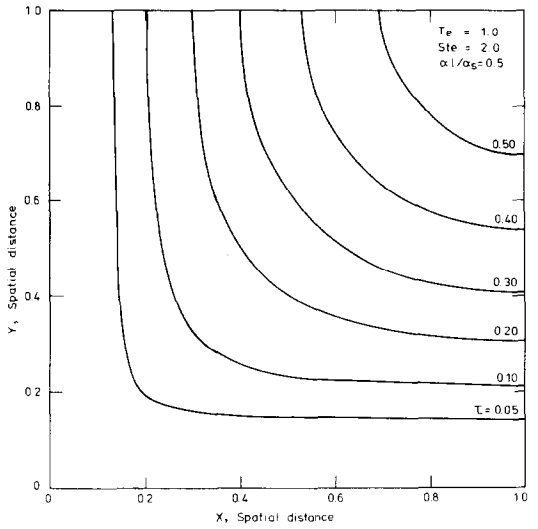

FIG. 6. Interface location at various times—square prism.

indicating the shape of the curve to be more flat on the diagonal. However, as the analytical results of Rathjen and Jiji [18] and the numerical results of Crowley [3] agree well with the present solution, the more appropriate shape of the interface would appear to be squarer as shown in Fig. 2.

When the liquid is initially at a temperature higher than its phase change temperature, the shape of the interface becomes more flat on the diagonal. This is indicated in Fig. 4 for which the operating parameters are $Ste = 2.0$, $\alpha_l/\alpha_s = 1$ and $T_e = 1.0$. These curves indicate that the liquid around the diagonal loses its heat faster than that situated away from the diagonal because of the predominant 2-D heat flow near the corner. The total solidification time is $\tau = 0.41$. Figure 5 shows the temperature histories of the region along two nodal planes. When the thermal diffusivity of the liquid is taken as half of that of the solid, no major change in shape of the interface is observed as shown in Fig. 6. The liquid temperature decreases less rapidly increasing the total solidification periods to $\tau = 0.61$.

Figure 7 shows the interface time histories for a rectangular prism undergoing phase change with the same operating parameters. Due to a decrease in the cross-sectional area of the prism the total solidification time is reduced to $\tau = 0.24$. The figure indicates that the interface on the adiabatic plane $Y = 0.7$ moves faster compared to the plane $X = 1$ because of their relative distances from the cold walls.

All the computations are performed on an IBM 370/155 computer. On average the problem without superheat took around 70 s, while the one with superheat needed around 120 s for the complete freezing of the region. Compared to the prevailing numerical methods, the present method gives considerable saving in computer time. The effect of time step size on the results obtained is studied by giving different time steps. The values obtained showed no significant change indicating the results to be independent of time step size.

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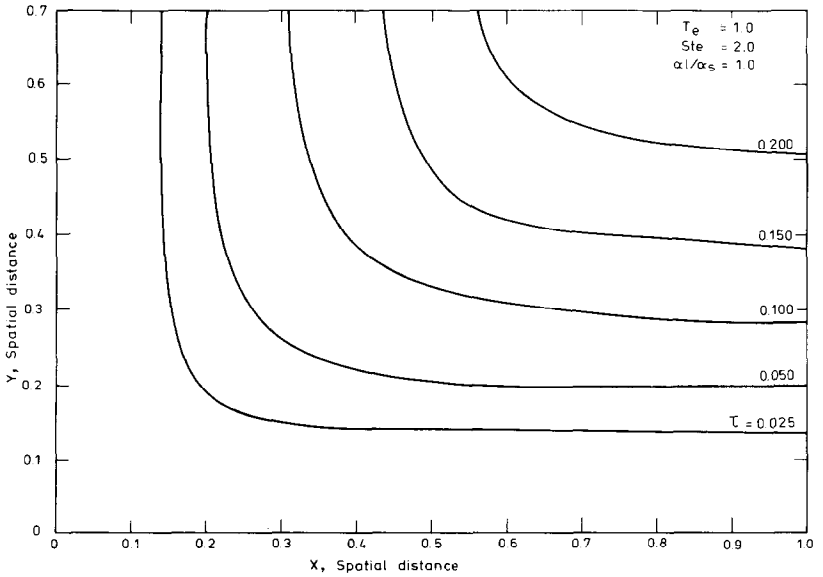

FIG. 7. Interface location at various times—rectangular prism.

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METHODE NUMERIQUE PERFORMANTE POUR LES PROBLEMES BIDIMENSIONNELS DE CHANGEMENT DE PHASE

Résumé—Le bilan précis de la méthode implicite aux directions alternées à balayage horizontal et vertical est efficacement incorporé dans le schéma numérique pour des problèmes bidimensionnels avec changement de phase. La méthode isole la non-linéarité associée à l'interface mobile et suit avec précision le mouvement de l'interface suivant les deux axes de coordonnées. Ceci évite les itérations aux noeuds ordinaires écartés de l'interface. Des résultats numériques sont obtenus et tracés pour des prismes carrés et rectangulaires subissant un changement de phase.

EIN EFFIZIENTES NUMERISCHES VERFAHREN FÜR ZWEIDIMENSIONALE PHASENÄNDERUNGSPROBLEME

Zusammenfassung—Der verfeinerte Ausgleich der horizontalen und vertikalen Abweichungen der Standard-ADI-Methode wird wirksam auf das numerische Verfahren für zweidimensionale Phasenänderungsprobleme übertragen. Die vorgestellte Methode trennt die Nichtlinearität ab, die mit der sich bewegenden Phasengrenzfläche verbunden ist, und führt die Bewegung der Phasengrenze in beiden Koordinatenrichtungen nach. Dieses Vorgehen vermeidet Iterationen an gewöhnlichen Knotenpunkten, die nicht an der Phasengrenze liegen. Numerische Ergebnisse werden ermittelt und gezeichnet für quadratische und rechteckige Prismen, die einem Phasenwechsel unterliegen.

ЭФФЕКТИВНЫЙ ЧИСЛЕННЫЙ МЕТОД ДЛЯ ДВУМЕРНЫХ ЗАДАЧ С ФАЗОВЫМ ПЕРЕХОДОМ

Аннотация—Наилучшие соотношения между горизонтальной и вертикальной развертками стандартного неявного метода переменных направлений эффективно использовано в численной схеме решения двумерных задач с фазовым переходом. Предлагаемый метод исключает нелинейность, связанную с движением границы раздела, и позволяет точно прослеживать ее движение вдоль обеих координатных осей, в результате чего отпадает необходимость проводить итерации в обычных узловых точках вдали от границы. Получены численные результаты, на основании которых составлены графики для квадратных и прямоугольных призм, претерпевающих фазовые превращения.